Student Project: Defect Conformal Field Theory

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These notes, written as a student project for the M2 ICFP Conformal Field Theory course at ENS Paris, provide an elementary introduction to defects in conformal field theory in $d > 2$ dimensions. We start with a general discussion regarding defect conformal field theory, including constraints on correlation functions and the bulk-defect OPE [1]. In particular, the bulk-defect OPE lies at the crux of a classic result in defect CFTs, namely the absence of non-trivial unitary defects in local free field theory [2]. Upon outlining the main arguments of the corresponding proof, we close this discussion with two examples of non-trivial defects in an interacting $O(N)$-symmetric bulk close to four dimensions [3–6].

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1 Introduction and summary

1.1 Introduction

Compared to arbitrary quantum field theories, the study of conformal field theories if often more tractable due to the added constraint of invariance under scale and special conformal transformations. Indeed, contrary to the case of the former, in CFT two and three-point functions have a fixed behavior up to structure constants referred to as CFT data, while the operator product expansion (OPE) allows one to express all higher-point functions using these constants [7]. These CFT data can in turn be approximated using the conformal bootstrap approach, which leverages symmetries and consistency conditions such as the associativity of the OPE [8], allowing one to approach solving a given theory.

While conformal symmetry isn't realized in nature *per se*, the study of CFTs can nonetheless yield invaluable insights into quantum field theory and gravity. On the one hand, QFTs can be viewed as renormalization group flows between ultraviolet and infrared fixed points, which are scale and often conformally invariant [9]. This program consists in exploring the underlying theory space by studying conformally invariant fixed points, and subsequently probing their behavior when they are perturbed by relevant operators triggering an RG flow to an IR fixed point. On the other hand, holographic principles, such as the AdS/CFT correspondence [10], have demonstrated that CFTs on a boundary of a given spacetime can be dual to a theory of gravity in the bulk of the same spacetime. Moreover, the physics on the worldsheet of a string is conformally – even Weyl – invariant, and all of this suggests that studying CFT allows one to obtain some insights into quantum gravity.

As it turns out, the picture of theory space as a plethora of conformally invariant fixed points and their corresponding universality classes isn't the full story. Indeed, suppose we start with a UV CFT in d dimensions, and then deform it using a relevant operator triggering an RG flow to an IR fixed point. Let's assume this operator is taken to be an integral $g\int d^p\tau\mathcal{O}(\tau)$ over a p-dimensional submanifold, for some coupling g and a local operator $\mathcal{O}(\tau)$. Both this operator and the submanifold on which it is supported are referred to as a *defect*. In principle, such a defect can break any number of symmetries, spacetime and internal, including conformal symmetry at the IR fixed point. However, as we shall see in section 4, there exist operators – conformal defects – which break conformal invariance in a *minimal* way which is discussed in section 2.3, giving rise to fixed points which are not CFTs, but instead invariant under the subgroup $SO(p+1,1)\times SO(d-p)$ of the full conformal group $SO(d+1,1)$[1]. Such theories are referred to as defect CFTs, or DCFTs for short. Hence, defects are very useful probes in quantum field theories [2], in that they allow us to reach more fixed points which are otherwise inaccessible via relevant bulk deformations of the form $g\int d^dx\mathcal{O}(x)$.

Beyond the role of defects as tools for exploring an abstract theory space [11, 12], there are a few model-dependent reasons one might be inclined to study these objects. For example, Wilson and 't Hooft operators, which are gauge invariant quantities in Yang-Mills theory, are famous examples of defects. In particular, Wilson loops are known to be good candidates for probing confinement [13, 14]. The study of extended objects in a higher dimensional bulk is also relevant to quantum gravity: D-branes and their intersections, now considered a vital component of the string theoretic landscape, are examples of defects [15–17]. Furthermore, viewed as the endpoints of an open string, D-branes play the role of a boundary in a two-dimensional worldsheet CFT, and boundaries are nothing but codimension-1 defects. The AdS/CFT correspondence also provides an example where boundaries or defects are highly relevant [10]. In addition to the usual statement of the correspondence, non-local CFTs can typically be viewed as living on the boundary of an AdS space where a local action is defined [18]. At lower energies, and more relevant to condensed matter physics, vortices, magnetic impurities and physical boundaries can all be viewed as concrete realizations of the abstract defects presented here [3, 19, 20]. Lastly, quantum entanglement is also an area where defects can serve an important purpose, since they have can be used to compute the dependence of entanglement

[1]Strictly speaking, the identification is between Lie algebras, but the shorthand $SO(d+1,1)$ for the conformal group is standard. Note that the $p=1$ case leads to $SL_2(\mathbb{R})\times SO(d-1)$ [4], while for $p=2$ Virasoro symmetry is not required on the defect, since the Virasoro algebra cannot be viewed as being included in the unbroken conformal algebra.

entropy on the shape of entangling surfaces [21].

1.2 Summary

Upon breaking conformal invariance with a defect, the first issue to address is how correlation functions are affected. Indeed, one might be worried that the loss of some of the constraints of a full-fledged CFT lead to a significant loss of structure in the form of correlation functions. As it so happens, defect CFTs still have correlators which are fully constrained up to *defect* CFT data, and the OPE again allows one to compute any higher-point function using lower-point data. Using the embedding space formalism [22], reviewed in sections 2.1 and 2.2, we derive the shape of some notable correlation functions which are specific to defect CFT. In particular, bulk primary operators can now be endowed with a physical one-point function, which is related to the distance between the corresponding operator insertion and the defect. For example, a bulk primary scalar ϕ of scaling dimension Δ has a one-point function

$$\langle \phi(x) \rangle = \frac{a_\phi}{|x|_\perp^\Delta} \,, \tag{1.1}$$

which can be viewed as a two-point function of the bulk operator with the defect. Moreover, the two-point function of a bulk primary scalar ϕ_1 of dimension Δ with a defect primary $\hat{\phi}_1$ scalar of dimension $\hat{\Delta}$ is given by [2]

$$\langle \phi_1(x_1) \hat{\phi}_2(x_2) \rangle = \frac{1}{|x_1 - x_2|^{2\hat{\Delta}} |x_1|_\perp^{\Delta - \hat{\Delta}}} \,, \tag{1.2}$$

which can be viewed as the three point-function of the defect operator, the bulk operator, and its mirror image by the defect[3].

In addition to conformal constraints, we also motivate the existence of a so-called *bulk-defect* OPE [23, 24], upon reviewing the usual arguments which lead to the OPE in a usual CFT in sections 2.6 and 2.7. More specifically, the bulk-defect OPE is an expansion in defect operators $\hat{\mathcal{O}}(x)$ of a bulk operator $\mathcal{O}(x)$ close to the defect:

$$\mathcal{O}(x) = \sum_k C_k(|x|_\perp, \partial) \hat{\mathcal{O}}_k(x_{||}) \,, \tag{1.3}$$

where we separate coordinates into components $x_\perp$ transverse to the defect and $x_{||}$ parallel to it. As in the case of an ordinary CFT, the bulk-defect OPE, combined with the usual OPE confined to the bulk or on the defect, are subject to consistency conditions, which lead to a corresponding defect bootstrap program [25–35]. In this light, one might want to classify all consistent defect in a given bulk theory, starting with a free bulk. Under additional assumptions, such as unitarity and locality, it turns out that all defects in a free theory are trivial [2], meaning that they are generalized free field theories, or Gaussian. The proof amounts to a) using the equations of motion of a local free field to constrain the defect operators that can appear in a bulk defect-OPE, i.e. constraining all defect operators, b) imposing consistency and analyticity on three-point functions involving the defect operators, which imply that defect scaling dimensions can only be of a certain form, and c) showing that this specific form entails that correlators of defect operators obey Wick's theorem. Some details are discussed in section 3, although we refer the reader to [2] for the complete argument.

Since the argument described above depends on the equations of motion being those of a free field, it breaks down in the case of an interacting bulk. As such, we close this short overview with two examples of interacting theories which have non-trivial defect dynamics in section 4, namely the localized magnetic field (or magnetic line defect) and surface defect in the $O(N)$ model, respectively given by the following actions:

[2]We use hats to denote defect quantities, as is standard in the literature.

[3]This statement should not be taken too literally in the case of a line defect. Take the mirror image to mean some other operator at the same distance to the defect.

$$S[\phi]=\frac{1}{2}\int d^dx\partial_\mu\phi_a\partial^\mu\phi^a+\frac{\lambda_0}{4!}\int d^dx\left(\phi_a(x)^2\right)^2+h_0\int d\tau\phi_1(\tau)\,,\tag{1.4}$$

$$S[\phi]=\frac{1}{2}\int d^dx\partial_\mu\phi_a\partial^\mu\phi_a+\frac{\lambda_0}{4!}\int d^dx\left(\phi_a(x)^2\right)^2+h_0\int d\tau\phi_a(\tau)^2\,,\tag{1.5}$$

where $a=1,\dots,N$ are the $O(N)$ indices implicitly summed over. These theories have non-trivial fixed points close to four dimensions, which are defect CFTs. In the localized magnetic field described by the action 1.4, we derive the following non-trivial defect CFT data close to the fixed point:

$$\hat{\Delta}_{\hat{\phi}}=1+\varepsilon-\varepsilon^2\frac{3N^2+49N+194}{2(N+8)^2}+O(\varepsilon^3),\quad a^2_{\phi}=\frac{N+8}{4}+\frac{2(N+8)^2\log 2+N^2-3N-22}{8(N+8)}\varepsilon+O\left(\varepsilon^2\right)\,,\tag{1.6}$$

while, in the surface defect described by the second action 1.5, we arrive at

$$\hat{\Delta}_{\hat{\phi}^2}=2+\frac{6}{N+8}\varepsilon+O(\varepsilon^2),\quad a_{\phi^2}=-\frac{3}{N+8}\sqrt{\frac{N}{2}}\varepsilon+O(\varepsilon^2)\,.\tag{1.7}$$

2 Elements of defect conformal field theory

2.1 The embedding space formalism

The conformal algebra in d dimensions coincides with that of $\mathfrak{so}(d+1,1)$. This strongly hints that we can somehow embed physical space into a larger bulk Minkowski space where conformal transformations become the familiar Lorentz transformations [1, 22]. This point of view – referred to as the *embedding space formalism* – is particularly useful, in that the former sometimes act nonlinearly – such is the case of the special conformal transformation – while the latter are always linear. Since the lightcone is Lorentz-invariant, let's pick lightcone coordinates (P_+,P_-,P^a) and use the metric η_{AB} such that

$$P\cdot P=-P_+P_-+\delta_{ab}P^aP^b\,,\tag{2.1}$$

picking Euclidean signature in physical space. The embedding from $\mathbb{R}^d$ to the lightcone of $\mathbb{R}^{d+1,1}$ is the following:

$$x\mapsto\mathbb{R}^*_+(1,x^2,x^\mu)\,,\tag{2.2}$$

i.e. we map points in physical space to rays on the lightcone in $\mathbb{R}^{d+1,1}$. This ensures that dimensions coincide and that the mapping is invertible. Next, we build a correspondence between fields on $\mathbb{R}^{d+1,1}$ which are $SO(d+1,1)$ tensors $F_{A_1\dots A_l}(P)$, and fields on $\mathbb{R}^d$, $f_{a_1\dots a_l}(x)$. In order to constrain F, we impose the following conditions, which will ensure that f transforms as a primary field:

1. F is defined on the cone $P^2=0$;
2. It is homogeneous of degree $-\Delta$ i.e. $\forall\lambda>0, F_{A_1\dots A_l}(\lambda P)=\lambda^{-\Delta}F_{A_1\dots A_l}(P)$;
3. It is symmetric and traceless, in particular for any index positions i,j, $\eta^{A_iA_j}F_{\dots A_i\dots A_j\dots}(P)=0$;
4. It is transverse: $(P\cdot F)_{A_2\dots A_l}=P^AF_{AA_2\dots A_l}=0$.

By homogeneity, we can restrict ourselves to the study of F on the Poincaré section

$$\{P_x=(1,x^2,x^a),x\in\mathbb{R}^d\}\,,\tag{2.3}$$

whose elements are in bijection with those of the physical space $\mathbb{R}^d$. Note that $SO(d+1,1)$ then simply rotates null rays into each other, thus sweeping the entire lightcone. Once F is determined, we can project it onto the physical field f via the following pull-back[4]:

[4] We sometimes drop x indices on the lightcone coordinates P for lighter notation.

$$f_{a_1\ldots a_l}(x)=\frac{\partial P^{A_1}}{\partial x^{a_1}}\ldots\frac{\partial P^{A_l}}{\partial x^{a_l}}F_{A_1\ldots A_l}(P_x)\,. \tag{2.4}$$

where we use a convention from [22] according to which capital letters indicate embedding space fields, while lowercase letters refer to fields in physical space. Thus defined, it can be shown that $f_{a_1\ldots a_k}$ corresponds exactly to a primary operator of spin l, i.e. under a conformal transformation $x\to x'$ of scale factor Ω [5]:

$$f'_{b_1\ldots b_l}(x')=\Omega^{l-\Delta}\frac{\partial x^{a_1}}{\partial x'^{b_1}}\ldots\frac{\partial x^{a_l}}{\partial x'^{b_l}}f_{a_1\ldots a_l}(x) \tag{2.5}$$

This projection satisfies two important properties. First, any tensor proportional to a lightcone coordinate P projects to zero. Indeed:

$$\frac{\partial P^A}{\partial x^a}P_A=\frac{1}{2}\frac{\partial}{\partial x^a}P^2=0\,. \tag{2.6}$$

Such tensors are referred to as *pure gauge* terms in the literature. Perhaps unsurprisingly, it can also be shown that two fields taken as above F and F' project to the same f if and only if they differ by a pure gauge. The second important property is that this projection maps traceless transverse tensors to traceless ones. Let's check this as well. Define the quantity

$$K^{AB}=\delta^{ab}\frac{\partial P^A}{\partial x^a}\frac{\partial P^B}{\partial x^b}\,. \tag{2.7}$$

The trace of the physical tensor is given by

$$\delta^{ab}f_{ab}=\delta^{ab}\frac{\partial P^A}{\partial x^a}\frac{\partial P^B}{\partial x^b}F_{AB}=K^{--}F_{--}+2K^{-,b}F_{-,b}+K^{ab}F_{ab}=4x^2F_{--}+4x^bF_{-,b}+\delta^{ab}F_{ab}\,. \tag{2.8}$$

Since F is transverse:

$$P^AF_{A-}=F_{+-}+x^2F_{--}+x^aF_{a-}=0\,, \tag{2.9}$$

which, subbed into the previous expression (2.8), gives the desired result:

$$\delta^{ab}f_{ab}=-4F_{+-}+\delta^{ab}F_{ab}=\eta^{AB}F_{AB}=0\,, \tag{2.10}$$

since F is traceless[6]. This shows that the embedding space fields we defined above will always project to physical tensors which are symmetric and traceless, as is usually required of spin-l representations.

Therefore, upon defining Lorentz-invariant quantities in embedding space up to pure gauge terms, projecting back to physical space using equation (2.4) will yield quantities, such as correlators, which transform appropriately under the conformal group.

2.2 Constraints on correlation functions

Now that we've introduced the machinery above, let's apply it to CFT correlators. Let's start with the trivial example of a one-point function. Consider a primary spin-l operator $f_{a_1\ldots a_l}(x)$. In embedding space, the only Lorentz-invariant quantity we can write is

$$\langle F(P)\rangle\propto P^2 \tag{2.11}$$

which trivially vanishes.

[5]There is a more general result for a rank (n,p)-tensor, we refer the reader to [36] for a pedagogical treatment.

[6]Note that the final equality is derived given $\eta^{+-}=-4$ (simple inversion of the 2×2 matrix representing the $+,-$ sector of the metric).

Consider the two point function of two primary scalars ϕ_1, ϕ_2. In embedding space, this amounts to looking at $\langle \Phi_1(P_1)\Phi_2(P_2)$ where ϕ_i is the projection of Φ_i and $P_i := P_{x_i}$. The embedding correlator must be a Lorentz scalar, hence it is of the form:

$$\langle \Phi_1(P_1)\Phi_2(P_2)\rangle = C\,(-2P_1 \cdot P_2)^{-\Delta} \tag{2.12}$$

where C is some constant, $\Delta > 0$ and the -2 factor is for convenience. Since Φ_i are individually homogeneous of degree $-\Delta_i$, we must have for any $\lambda_i > 0$:

$$C = 0 \quad \text{or} \quad \lambda_1^{-\Delta_1}\lambda_2^{-\Delta_2} = (\lambda_1\lambda_2)^{-\Delta} \tag{2.13}$$

i.e. if $C \neq 0$, $\Delta_1 = \Delta_2 = \Delta$. Thus the two-point function above is non-zero if and only if the two primaries have the same scaling dimension. Note that this argument is much simpler than using special conformal transformations. The last ingredient is the identity

$$-2P_1 \cdot P_2 = -2\left(-\frac{1}{2}x_1^2 - \frac{1}{2}x_2^2 + \delta_{ab}x_1^a x_2^b\right) = |x_1 - x_2|^2 \,. \tag{2.14}$$

Projecting back into physical space, for two primaries of same conformal weight Δ:

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle \propto \frac{1}{|x_1 - x_2|^{2\Delta}} \,. \tag{2.15}$$

Let's keep going with the three-point function. It must be a Lorentz scalar, hence of the form:

$$\langle \Phi_1(x_1)\Phi_2(x_2)\Phi_3(x_3)\rangle = C(-2P_1 \cdot P_2)^{-\Delta_{12}}(-2P_1 \cdot P_3)^{-\Delta_{13}}(-2P_2 \cdot P_3)^{-\Delta_{23}} \,. \tag{2.16}$$

For $C \neq 0$, homogeneity requires for any $\lambda_i > 0$:

$$\lambda_1^{-\Delta_1}\lambda_2^{-\Delta_2}\lambda_3^{-\Delta_3} = (\lambda_1\lambda_2)^{-\Delta_{12}}(\lambda_1\lambda_3)^{-\Delta_{13}}(\lambda_2\lambda_3)^{-\Delta_{23}} \,, \tag{2.17}$$

which can hold only if and only if

$$\begin{cases} \Delta_{12} + \Delta_{13} = \Delta_1 \\ \Delta_{12} + \Delta_{23} = \Delta_2 \\ \Delta_{13} + \Delta_{23} = \Delta_3 \end{cases} \tag{2.18}$$

Solving this system of equations, we arrive at the well-known expression:

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = \frac{\lambda_{123}}{|x_1 - x_2|^{\Delta_1+\Delta_2-\Delta_3}|x_1 - x_3|^{\Delta_1+\Delta_3-\Delta_2}|x_2 - x_3|^{\Delta_2+\Delta_3-\Delta_1}} \,. \tag{2.19}$$

As usual, four-point functions aren't fully constrained, since we can form conformal cross-ratios in embedding space:

$$u = \frac{P_{12}P_{34}}{P_{13}P_{23}}, \quad v = \frac{P_{14}P_{23}}{P_{13}P_{24}} \,. \tag{2.20}$$

This method is extremely powerful for deriving general forms of correlation functions. However, it is perhaps unnecessary for scalar primaries since they are not incredibly hard to derive in physical space either. Notwithstanding, a CFT usually has primary vectors, tensors and general spinning operators. Handling those cases is less obvious than in 2D CFT, where spin is merely a consequence of including holomorphic and antiholomorphic sectors. The embedding space formalism becomes a clear favorite for deriving their correlators. For example, let's derive the two-point function of two vectors $v_1(x_1), v_2(x_2)$. The two-point will be transverse by virtue of the transversality of its constituent fields. Omitting pure gauge terms which are projected out, this forces us to write in embedding space:

$$\langle V_{1,A}(P_1)V_{2,B}(P_2)\rangle = C(-2P_1 \cdot P_2)^{-\Delta}\left(\eta_{AB} - \frac{P_{1B}P_{2A}}{P_1 \cdot P_2}\right) \,, \tag{2.21}$$

where we again use the common convention of having capital letters for embedding fields, and lowercase letters for physical ones. The tensor structure leaves the scaling dimension unaffected, hence we again have $\Delta = \Delta_1 = \Delta_2$ for non-vanishing primary vector two-point functions. Note that a pairing $P_{1A}P_{2B}$ in the second term would have been a pure gauge contribution projecting to zero. All that remains is to project the tensor structure. To do so, we need the covariant components of P:

$$P_{1B} = \eta_{BC}P_1^C = \left(-\frac{1}{2}x_1^2, -\frac{1}{2}, x_{1,a}\right) . \tag{2.22}$$

Therefore

$$\frac{\partial P_1^A}{\partial x_1^a}\frac{\partial P_2^B}{\partial x_2^b}P_{2A}P_{1B} = (-x_{1,a}+x_{2,a})(-x_{2,b}+x_{1,b}) = -(x_1-x_2)_a(x_2-x_2)_b\,. \tag{2.23}$$

The remaining steps are straightforward, and we arrive at

$$\langle v_{1,a}(x_1)v_{2,b}(x_2)\rangle = \frac{1}{|x_1-x_2|^{2\Delta}}\left(\delta_{ab} - \frac{2(x_1-x_2)_a(x_1-x_2)_b}{|x_1-x_2|^2}\right) = \frac{I_{ab}}{|x_1-x_2|^{2\Delta}}\,. \tag{2.24}$$

The same sort of reasoning applies to higher spin operators. Consider spin 2 for example. We can reuse the reasoning above, namely constructing a general transverse and symmetric tensor structure, and projecting. The additional sublety here is that we also have tracelesness relative to the indices of the constituent 2-tensors taken individually. This yields the following for spin 2:

$$\langle t_{a_1a_2}(x_1)t_{b_1b_2}(x_2)\rangle = \frac{const.}{|x_1-x_2|^{2\Delta}}\left[\frac{1}{2}\left(I_{a_1b_1}I_{a_2b_2}+I_{a_1b_2}I_{a_2b_1}\right) - \frac{1}{d}I_{a_1a_2}I_{b_1b_2}\right] , \tag{2.25}$$

where we reuse the tensor structure I_{ab} from equation (2.24). One might want to directly apply this to the stress tensor. However, as in 2D CFT it is not a primary. It is possible to encode the conservation of currents in embedding space, but we merely cite the final result below:

$$\langle T_{a_1a_2}(x_1)T_{b_1b_2}(x_2)\rangle = \frac{c}{|x_1-x_2|^{2d}}\left[\frac{1}{2}\left(I_{a_1b_1}I_{a_2b_2}+I_{a_1b_2}I_{a_2b_1}\right) - \frac{1}{d}\delta_{a_1a_2}\delta_{b_1b_2}\right] , \tag{2.26}$$

where the use of c is not an accident since is nothing but the central charge of the theory. While c is not present in the commutation relations of the conformal algebra in $d > 2$, it is still a physical quantity since the normalization of the stress tensor two-point function isn't free as for two-point functions of primaries.

One could in principle keep going with higher spin. However, it becomes tedious to work with tensors in index notation. Instead, the embedding space formalism is enhanced via the use of symmetric polynomials whose coefficients are the very tensors we are looking for. The constraints on embedding tensors translate to constraints on the aforementioned polynomials. Upon writing a correct polynomial, one can obtain the final result for correlators by applying so-called Todorov differential operators whose role is to extract their coefficients, namely the desired transverse traceless symmetric tensors [1, 22]. However, since we are not looking to classify correlators of arbitrary spin here, we won't be needing these techniques.

2.3 Prototypical example of a defect

Before applying the embedding space formalism to defect CFTs, let's take a step back and attempt to gain a better picture of what a defect is before proceeding with our study. A defect consists in a non-local operator which extends over a p-dimensional subspace of a given d-dimensional bulk, such that the following symmetry breaking takes place:

$$SO(d+1,1) \to SO(d-p)\times SO(p+1,1)\,. \tag{2.27}$$

A typical example of this, and one which is closely related to section 4, is ϕ^4 theory. Consider the following action in Euclidean space:

$$S_0 = \int d^d x \left[\frac{1}{2} \partial_\mu \phi(x) \partial^\mu \phi(y) + \frac{\lambda_0}{4!} \phi(x)^4 \right] . \tag{2.28}$$

This theory is classically conformally invariant in four dimensions. To promote this invariance to the quantum level, one can study its behavior under RG flow, for example via an ε-expansion which renders λ marginally relevant. This leads to the following β-function:

$$\beta_\lambda = -\varepsilon\lambda + \frac{3}{16\pi^2}\lambda^2 + O(\lambda^3) . \tag{2.29}$$

This has a non-trivial, Wilson-Fisher fixed point in the IR given by $\lambda_* = 16\pi^2\varepsilon/3$. The quantum theory is known to be conformally invariant at this point. Now let's break this conformal symmetry by adding a defect. Consider therefore the following action:

$$S = S_0 + h_0 \int d^p \tau \phi(\tau)^k , \tag{2.30}$$

where the integral is taken over a p-dimensional subspace. In principle, this can be any submanifold which is has a non-trivial conformal group. However, those that are usually studied are either flat or spherical. We restrict ourselves to the former in this study, although spherical defects are particularly useful for computing measurements of irreversibility under defect RG flow, such as the g-function or free energy [4, 37–39].

In order to have any hope for the theory confined to the defect to be invariant under the restricted conformal group $SO(p+1,1)$, it should at least be invariant classically. We must therefore require that h_0 be classically marginal:

$$k\Delta_\phi = p \tag{2.31}$$

i.e. since $\Delta_\phi = (d-2)/2$ for a (local) free scalar,

$$k = \frac{2p}{d-2} . \tag{2.32}$$

Hence, if we want to have a line defect CFT close to four dimensions, we must take $k = 1$, while for a surface defect we ought to take $k = 2$. If we want to work with a boundary, we must take $k = 3$, although we won't be studying cubic interactions on a boundary in this work (we refer the reader to [40]).

Our next order of business is to derive general forms for correlators in the presence of a defect. However, upon introducing a defect, we no longer have translation symmetry along the direction orthogonal to the defect, nor symmetry under rotations mixing the coordinates parallel and orthogonal to it. Hence, the results for the two-point and three-point functions derived previously are no longer valid. Nevertheless, relatively simple statements can still be made about correlators in some situations by adapting the formalism presented in sections 2.1 and 2.2.

2.4 Embedding of flat defects and constraints on correlators

For a flat defect $\mathcal{D}$, there is not much more we need to do when it comes to the embedding space formalism, apart from specifying where the defect is mapped on the lightcone. In physical space, let's define coordinates (x^a, x^i) such that the x^a correspond to coordinates parallel to the defect and x^i orthogonal to it. These conventions carry over to the lightcone [2]:

$$P^M \in \mathcal{D} : P^A = (1, x^2, x^a), \quad P^I = 0 . \tag{2.33}$$

Following [1], we define two distinct scalar products, which are involved in defining scalar quantities on the lightcone:

$$P \bullet Q = P^A \eta_{AB} Q^B , \quad P \circ Q = P^I \eta_{IJ} P^J . \tag{2.34}$$

In the presence of a defect, the usual definition of a primary operator, except that its transformation properties are now restricted to transformations taken in the subgroup $SO(p+1,1)\times SO(d-p)$. Moreover, primary operators are distinguished by whether they live in the bulk or on the defect. As is customary, we will write operators on the defect with hats to better distinguish them from bulk operators. Note that since the theory on the defect is $SO(p+1,1)$-invariant, correlators exclusively involving defect primaries satisfy the same expressions as those derived in traditional CFT. Notwithstanding, some new behavior appears when adding bulk operators to the mix.

Let's start with the one-point function. While this was perhaps a trivial example previously, it turns out that certain bulk operators acquire a non-vanishing expectation value when a defect is introduced. This can be viewed as a consequence of having introduced a new length scale, namely the distance between the operator insertion in question and the defect. Consider the one-point function of a bulk primary scalar of dimension Δ. Keeping notations from section 2.2, the only Lorentz scalar quantity that can be written in embedding space is

$$\langle \Phi(P)\rangle = C(-2P\bullet P)^{-\Delta_1}(-2P\circ P)^{-\Delta_2}\,. \tag{2.35}$$

for some constants Δ_1,Δ_2. Recall that P is still null, so we actually have $P\bullet P=-P\circ P$ i.e. the tangential and orthogonal norms aren't independent from one another. We therefore have the ansatz

$$\langle \Phi(P)\rangle = C(-2P\circ P)^{-\Delta_1-\Delta_2}\,. \tag{2.36}$$

Imposing homogeneity relative to P, we arrive at

$$\langle \phi(P)\rangle = \frac{A_\phi}{|x|_\perp^\Delta}\,. \tag{2.37}$$

To construct the one-point function of a bulk vector, we need to build a vector quantity out of P. The only possibility is to have an expression which is supported along directions orthogonal to the defect, and immediately taking into account homogeneity, we arrive at

$$\langle V_I(P)\rangle = C(-2P\circ P)^{-\Delta/2}P_I\,. \tag{2.38}$$

P_I contains only the last type of components (no $\pm$) due to the embedding of the defect defined in (2.33), hence:

$$\langle v_i(x)\rangle = \frac{A_v x_i}{|x|_\perp^\Delta}\,. \tag{2.39}$$

Note that this is parity-odd, therefore if we want our theory to be invariant under parity, this must vanish (i.e. A_v is set to zero). This result can be generalized to any odd spin bulk operator [2]. However, the story is different for spin-2 operators, which can acquire a VEV which does not break parity. For example, using the general reasoning above with additional calculations, one can derive the following VEV for the bulk stress tensor [36]:

$$\langle T_{ij}\rangle = \frac{h}{|x|_\perp^d}\left(J_{ij}-\frac{d-p-1}{d}\delta_{ij}\right)\,, \tag{2.40}$$

where

$$J_{ij} = \delta_{ij}-\frac{x_i x_j}{|x|^2}\,. \tag{2.41}$$

Let's move on to the two-point function. There are three cases to consider: defect-defect, bulk-defect and bulk-bulk correlators. The theory on the defect is a lower dimensional CFT with a global symmetry related to $SO(d-p)$, therefore up to quantum numbers associated with transverse rotations, we should get the same two-point functions as in a usual CFT. Let's examine the bulk-defect case. Consider two primary scalars ϕ_1 and $\hat{\phi}_2$ of dimensions Δ and $\hat{\Delta}$ respectively (recall hats indicate an operator living on the defect). The most general relation we can write in embedding space is

$$\langle \Phi_1(P_1)\hat{\Phi}_2(P_2)\rangle = C(-2P_1 \circ P_1)^{-\Delta_1}(-2P_1 \cdot P_2)^{-\Delta_{12}}\,. \tag{2.42}$$

for some constants Δ_1, Δ_{12}. Using homogeneity with respect to P^1 and P^2, we arrive at

$$\langle \phi_1(x_1)\hat{\phi}_2(x_2)\rangle = \frac{1}{|x_1 - x_2|^{2\hat{\Delta}}|x_1|_\perp^{\Delta-\hat{\Delta}}}\,. \tag{2.43}$$

Lastly, let's examine the bulk-bulk case. It turns out we can construct conformal cross ratios for the bulk two-point function [36],

$$\xi_1 = \frac{(x_1 - x_2)^2}{4|x_1|_\perp |x_2|_\perp}, \quad \xi_2 = \frac{x_1 \circ x_2}{|x_1|_\perp |x_2|_\perp}\,, \tag{2.44}$$

since we have additional length scales compared to the case without a defect. In short, the bulk-bulk two-point function behaves as a four-point function in usual CFT. There is a particularly nice way of interpreting these results. Indeed, whenever a bulk operator is inserted, one has to include its distance r from the defect. For example, the one-point function of an operator in the bulk no longer vanishes (it can always be made to vanish in a regular QFT using the redefinition $\mathcal{O} \to \mathcal{O} + const.$). The one-point function in defect CFT can be viewed as a usual two-point function of the bulk operator with the defect. As for the bulk-defect two-point function, it can be viewed as the three-point function of the bulk operator, the defect operator and the mirror image of the bulk operator by the defect, and so on and so forth for higher-point functions.

The next logical step is to look at the non-trivial correlation functions, and use consistency conditions on the operator product expansion (OPE) to constrain the OPE coefficients and solve, in a moderate sense and for all intents and purposes, the defect CFT in question. However, in light of the stress tensor appearing in this section, we briefly pause this discussion to address what exactly happens to the stress tensor in a theory with a defect, since it is one of the most important operators of any given local theory.

2.5 An aside on the stress tensor

Upon breaking $SO(d+1,1)$, including translation invariance along the direction orthogonal to a defect, one might wonder what happens to conservation of energy-momentum. Indeed, it is expected that the continuity equation is violated for transverse coordinates, and this is indeed the case. However, the "anomaly" associated with this violation of conservation turns out to yield a valuable operator in defect CFT: the displacement operator. Suppose the embedding of the defect in the bulk is given by some functions $X^m(\tau^a)$, $0 \le m \le d-1$, $0 \le a \le p-1$. Just as the stress tensor measures the response of a system to a change of metric, the displacement operator D_m can be defined as measuring the response to a change in the location of the defect. The full variation of the action is therefore

$$\delta S = \frac{1}{2}\int d^dx\sqrt{g}T_{mn}\delta g^{mn} - \int d^p\tau\sqrt{\gamma}D_m\delta X^m\,, \tag{2.45}$$

where γ_{ab} is the induced metric on the defect [36]. Suppose both the bulk and the defect are flat. Under a change of coordinates $x' = x + \varepsilon$,

$$\delta S = -\int d^dx T_{mn}\partial^m\varepsilon^n - \int d^p\tau D_m\varepsilon^m\,, \tag{2.46}$$

which entails the following on-shell:

$$\partial_m T^{mn} = \delta^{(d-p)}(x_\perp)D^n\,. \tag{2.47}$$

There is an additional symmetry we haven't used yet: reparametrization invariance of the defect. Under a change of parametrization of the defect, the embedding functions change by $\delta X^m = \partial_a X^m \delta\tau^a$, which entails that $D_m\partial_a X^m = 0$ since $\delta S = 0$ and $\delta g^{mn} = 0$ in the variation (2.45). Hence, as mentioned previously, conservation of energy-momentum is only violated in directions transverse to the defect, $\partial_a X^m$ being a vector tangent to the defect. As in usual CFT, the next step is to derive Ward identities, and see how they constrain correlation functions and OPEs now that the stress tensor comes with an additional operator D^m. We won't do this here however, and instead refer the reader to [1, 36].

Let's get back to our main discussion. Next, we review the OPE in $d > 2$ dimensions and in the presence of a defect.

2.6 Radial quantization and state-operator correspondence

As in two-dimensional CFT, the operator product expansion is integral to the study of the theory in question. The OPE can be viewed as stemming from so-called *radial quantization*. Suppose the CFT is defined in flat space endowed with the metric

$$ds^2 = dr^2 + r^2 d\Omega_{d-1}^2 \,. \tag{2.48}$$

Now consider the change of radial coordinate $r = e^\tau$. This is a conformal transformation leading to

$$ds^2 = e^{2r}\left(d\tau^2 + d\Omega_{d-1}^2\right) \,, \tag{2.49}$$

which is conformal to a cylinder $\mathbb{R} \times \mathbb{S}^{d-1}$. The real coordinate τ plays the role of time, and it can be associated with a generator of "time" translations. Recall the following relations from the conformal algebra in Euclidean signature:

$$[D, P_\mu] = P_\mu, \quad [D, K_\mu] = -K_\mu \,, \tag{2.50}$$

where P_μ is the generator of translations and K_μ that of special conformal transformations. These motivate the following identification: D can be viewed as a Hamiltonian generating time translations, while P_μ and K_μ as raising and lowering operators respectively. Viewing the theory on the cylinder as a quantum theory with τ as a direction of time is often referred to as radial quantization.

Now let $\mathcal{O}(0)$ be a primary operator inserted at the origin of $\mathbb{R}^d$. Recall that under a conformal transformation $x \mapsto x'$, we have:

$$\mathcal{O}'(x') = \left|\frac{\partial x}{\partial x'}\right|^{\frac{\Delta}{d}} \mathcal{O}(x) \,, \tag{2.51}$$

which translates to the following via the sphere-cylinder map described above:

$$\mathcal{O}'(\tau) = r^{\Delta/d}\mathcal{O}(r) \,. \tag{2.52}$$

Suppose that we have a Hilbert space on the cylinder on which the cylinder operators act, endowed with a conformally invariant vacuum state $|0\rangle$. We can then associate to the operator inserted at the origin of $\mathbb{R}^d$ the state $|\mathcal{O}\rangle = \mathcal{O}'(-\infty)|0\rangle$, and subsequently time evolve this state using exponentiation of the cylinder Hamiltonian D. Note that we can obtain eigenstates of D on the cylinder using primaries at the origin:

$$D\mathcal{O}(0) \cdot |0\rangle = [D, \mathcal{O}(0)] \cdot |0\rangle = \Delta|0\rangle \,, \tag{2.53}$$

and descendant operators will similarly yield descendant states.

Conversely, let $|\psi(t)\rangle$ be some state on the cylinder at some fixed time slice. The state can be propagated backwards in time until it corresponds to a small sphere around the origin in $\mathbb{R}^d$. Using rotations, this completely fixes how the operator we wish to construct behaves on this sphere. Hence, we are left with a boundary condition which fully constrains the operator $\mathcal{O}(0)$ we are after (this can be viewed as a consequence of scale invariance) [41]. This is the famous *state-operator correspondence* in CFT.

In the presence of a defect, it turns out that radial quantization and the state-operator correspondence are left intact, up to an appropriate choice of radial coordinates with the origin placed on the defect [23, 24]. As such, the state-operator correspondence is preserved in defect CFT.

2.7 Operator product expansion

The state-operator correspondence we motivated above paves the way to an operator product expansion in d dimensions. Let's start with the case of generic CFT. Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two local operators. Taking their product and acting on the vacumm yields a state. This state can subsequently be expanded in a basis of eigenstates of the dilation operator D:

$$\mathcal{O}_1(x)\mathcal{O}_2(0)|0\rangle = \sum_n c_n|\Delta_n\rangle\,. \tag{2.54}$$

Now, by state-operator correspondence, we know that $|\Delta_n\rangle$ is associated to some operator $\mathcal{O}_{\Delta_n}(0)$ of scaling dimension Δ_n acting on the vacuum. Moreover, since P_μ, K_μ raise and lower scaling dimensions, Δ_n is in fact associated with a multiplet containing a primary and its descendants [36]. Hence, the above expression can be recast in the following way:

$$\mathcal{O}_1(x)\mathcal{O}_2(0)|0\rangle = \sum_k C_{12}^k(x,\partial)\mathcal{O}_k(0)|0\rangle\,, \tag{2.55}$$

where $\mathcal{O}_k(0)$ are primary operators. Dropping the vacuum states, we arrive at the usual form of the OPE:

$$\mathcal{O}_1(x)\mathcal{O}_2(0) = \sum_n C_{12}^k(x,\partial)\mathcal{O}_k(0)\,. \tag{2.56}$$

As in 2D CFT, this can be constrained by symmetry (note that we've already implemented translation symmetry implicitly by the choice of coordinates for the local operators). We get a similar expression to the 2D case for the OPE coefficients to lowest subleading order by constraining the expansion via conformal transformations:

$$C_{12}^k(x,\partial) \propto \frac{\lambda_{12k}}{|x|^{\Delta_1+\Delta_2-\Delta_k}}\left(1+\frac{\Delta_1-\Delta_2+\Delta_k}{2\Delta_k}x^\mu\partial_\mu+\dots\right)\,. \tag{2.57}$$

Note that the ambiguity in the proportionality factor essentially stems from the normalization factor of two-point functions. Furthermore, it's worth mentioning the OPE of two operators is often viewed as being defined when they get arbitrarily close to one another. This is the correct point of view in a general QFT. Notwithstanding, in CFT the radius of convergence of the expansion is often finite, provided we work in a ball containing the two operators in question and no others.

Since radial quantization and the state-operator correspondence also hold in defect CFT, we can recover an OPE in the presence of a defect. However, as in our study of correlation functions, we have more cases to consider when adding a defect. Assuming we have a notion of locality and a cluster decomposition (essentially statistical independence between operators taken far apart), we can recover the OPE described above. Indeed, in the presence of a defect, the OPE of two bulk or defect operators taken close together is essentially unaffected. There are some non-trivial assumptions behind this statement that we won't address here [36]. There is a new type of OPE, however, where a local operator is taken close to the defect. Indeed, such an operator $\mathcal{O}(x)$ can then be reproduced by an expansion in defect operators [2, 36]:

$$\mathcal{O}(x) = \sum_k C_k(|x|_\perp,\partial)\hat{\mathcal{O}}_k(x_{||})\,, \tag{2.58}$$

One way to motivate this is that, under defect radial quantization, we've placed the origin on the defect, and hence operators $\mathcal{O}(0)$ are replaced by defect operators $\hat{\mathcal{O}}(0)$.

Since the bulk two-point function in the presence of a defect behaves as a four-point function without a defect, one might guess that using the OPE with a bulk-bulk two-point function here might lead to bootstrap equations. This is indeed the case. The two-point function can be rewritten either by taking the expectation value of the bulk OPE of two fields, or by taking that of the product of two bulk-defect OPEs. We hence have two ways of writing the same thing, which constrains the values OPE coefficients may take. This concludes our tour of $d > 2$ CFT and defects.

3 Interlude: non-trivial defects in free field theory?

3.1 Overview

Before moving on to concrete realizations of defect CFTs, it is instructive to address whether there may even exist non-trivial unitary defects in a given bulk theory. Doing this in general is highly non-trivial, and is related to the defect bootstrap program [25–35]. However, it has been shown that in integer dimensions less than four, local theories which are free in the bulk cannot have non-trivial defects [2] – trivial meaning defect correlators obey Wick's theorem just as in the case of a Gaussian theory[7]. As the statement of the theorem suggests, there are two loopholes:

- **Bulk interactions**: An obvious exception is adding bulk interactions. This is the case for ϕ^4 theory, for which there are known defect theories which will be presented in what follows [3–6] This will be the case for example in the localized magnetic field examined in section 4.1.
- **Non-locality**: A free scalar field is usually taken to be local, i.e. its action is written as an integral of Lagrangian density. However, if the Lagrangian itself is an integral of another density, the theory is non-local. One famous example of a non-local field theory is the long-range Ising model, which has been extensively studied from a CFT perspective [18, 42–44]. It loosely amounts to replacing the Laplacian by a fractional Laplacian in ϕ^4 theory. Recently, there has been some headway in building defects in the long-range Ising model, and it has been shown that non-local free field theory can have non-trivial unitary defects [45].

In the following, we very roughly sketch out the proof of triviality presented in [2]. The crux of the argument relies on constraining the bulk-defect OPEs of bulk fields, which subsequently yield constraints on the defect operators. Next, using consistency and analyticity arguments, one can show that correlators of the defect fields obey Wick's theorem. To simplify notations in keeping with [2], we work in codimension $q = d - p = 2$, although the argument is generalized in the same reference.

3.2 Two-point function of the free scalar

Let ϕ be a scalar bulk primary and $\hat{\mathcal{O}}_{-s}$ a defect primary operator of transverse spin s (transverse to the defect that is). Since we're working in codimension $q = 2$, we can use complex notation $(z, \bar{z})$ for the transverse directions, and $\vec{x}$ for the directions parallel to the defect. The two-point function of these two operators is given by [1, 2]:

$$\langle \phi(\vec{x}, z, \bar{z}) \hat{\mathcal{O}}_{-s}(0) \rangle = \frac{b_\phi^{\hat{\mathcal{O}}} \bar{z}^s}{|z|^{\Delta_\phi - \hat{\Delta}_{\hat{\mathcal{O}}} + s} (|z|^2 + |\vec{x}|^2)^{\hat{\Delta}_{\hat{\mathcal{O}}}}} , \tag{3.1}$$

where we use notations identical to those used in the aforementioned references. Now, suppose the field ϕ obeys a Klein-Gordon equation (this excludes non-local theories, which turn out to have non-trivial defects [45]). Now, acting with the d'Alembertian away from contact points:

$$0 = \langle \Box \phi(\vec{x}, z, \bar{z}) \hat{\mathcal{O}}_{-s}(0) \rangle \sim \left(\hat{\Delta}_{\hat{\mathcal{O}}} - \Delta_\phi + |s| \right) \left(\hat{\Delta}_{\hat{\mathcal{O}}} - \Delta_\phi - |s| \right) \frac{b_\phi^{\hat{\mathcal{O}}} \bar{z}^s}{|z|^{2 + \Delta_\phi - \hat{\Delta}_{\hat{\mathcal{O}}} + s} (|z|^2 + |\vec{x}|^2)^{\hat{\Delta}_{\hat{\mathcal{O}}}}} , \tag{3.2}$$

which entails that the only defect primary operators allowed in the bulk-defect OPE of a free scalar must satisfy $\hat{\Delta}_s^\pm = \Delta_\phi \pm |s|$. We denote such operators $\psi_s^\pm$. In a unitary defect CFT, the following conditions must also be met:

$$\begin{aligned} &\hat{\Delta} \geq \frac{p}{2} - 1 \text{ or } \hat{\Delta} = 0, \quad && \text{if } p > 2, \\ &\hat{\Delta} \geq 0, && \text{if } p \geq 2 \,. \end{aligned} \tag{3.3}$$

[7] This not the full statement from [2], but it will be enough for our purposes and for justifying the counter-examples studied in section 4.

This significantly constrains which $\psi_s^\pm$ can exist, since we cannot have a spectrum unbounded from below. In particular, this rules out ψ_s^- for $s > 1$ since they have dimension $p/2 - s$. For additional technical reasons, addressing the monodromy of the defect among other things, we arrive at the following bulk-defect OPE:

$$\phi(\vec{x}, z, \bar{z}) = \sum_{\substack{s\geq 0 \\ s\in\mathbb{Z}/2}} \left(b_\phi^{+,s} \bar{z}^s C_{\Delta_\phi + s}(|z|, \partial) \psi_s^+(\vec{x}) + c.c. \right) . \tag{3.4}$$

This completely fixes the two-point function of the bulk field ϕ and tells us what kind of operator can exist on the defect. However, additional arguments are required for higher-point functions. We essentially need the defect OPE of the ψ_s^+.

3.3 Defect OPE of the ψ_s^+

In order to understand the dynamics on the defect, it is crucial to know the OPE of the ψ_s^+ operators. Since the full argument is lengthy, we merely summarize the strategy here. The idea is to consider two families of three-point functions, namely bulk-defect-defect and bulk-bulk-defect three-point functions. At first glance, the first type of correlator contains singularities, which nevertheless turn out to be unphysical and can even be removed provided special conditions are met. As for the second case, requiring consistency of the bulk-defect and bulk-bulk OPEs provides further special requirements.

Bulk-defect-defect Let ϕ be a bulk operator, $\hat{\mathcal{O}}$ an $SO(p)$ scalar and $\hat{T}$ a symmetric traceless tensor of parallel spin j. Using the bulk-defect OPE and requiring analyticity leads to:

$$\hat{\Delta}_{\hat{\mathcal{O}}} = \hat{\Delta}_s^+ + \hat{\Delta}_{\hat{T}} + j + 2n, \quad n \in \mathbb{N} . \tag{3.5}$$

This is referred to as a *double-twist* combination of the dimensions of $\hat{T}$ and ψ_s^+.

Bulk-bulk-defect In keeping with notations from the bulk-defect-defect case, one then looks at $\langle \phi\phi\hat{T} \rangle$. This correlator can be expanded using the bulk-bulk OPE of the ϕ fields, or the bulk-defect OPE of each ϕ field individually. Requiring the consistency of these two approaches leads to the following double-twist condition:

$$\hat{\Delta}_{\hat{T}} = \hat{\Delta}_{s_1}^+ + \hat{\Delta}_{s_2}^+ + j + 2n, \quad n \in \mathbb{N} . \tag{3.6}$$

The two cases above therefore lead to the following form for $\psi\psi$ OPEs:

$$\psi_{s_1} \times \psi_{s_2} = \delta_{s_1 s_2} \mathbf{1} + \{\text{operators with twist } \hat{\Delta}_{s_1} + \hat{\Delta}_{s_2} + 2k, k \in \mathbb{N}\} . \tag{3.7}$$

It turns out that this is enough to show that all n-point functions of the ψ_s are those of generalized free fields i.e. obeying Wick's theorem [2]. This concludes the argument.

Now that we have some general idea of what a defect is and under which circumstances it might yield interesting dynamics, let's look at some famous examples in scalar field theory.

4 Defects in the $O(N)$ model

4.1 Localized magnetic field in the $O(N)$ model

4.1.1 Presentation of the model

The localized magnetic field is the theory of a line defect in a bulk supporting interacting scalar fields, given by the following action:

$$S[\phi] = \frac{1}{2}\int d^dx\partial_\mu\phi_a\partial^\mu\phi^a + \frac{\lambda_0}{4!}\int d^dx\left(\phi_a(x)^2\right)^2 + h_0\int d\tau\phi_1(\tau)\,, \tag{4.1}$$

where the a indices implictly summed over are $O(N)$ indices in the defining representation. The name of this model can be motivated by the fact that it is the continuum, coarse-grained analogue of the following lattice model:

$$H[\{\vec{S}_i\}] = -J\sum_{\langle i,j\rangle}\vec{S}_i\cdot\vec{S}_j + h\sum_i S_{i,1}\,, \tag{4.2}$$

which is nothing but the nearest-neighbor $O(N)$ model in a constant magnetic field in the $a=1$ direction. The continuum model was extensively studied in [3, 4, 46].

4.1.2 ε-expansion

In order to have a conformally invariant quantum field theory, we need to figure out what the fixed point of the above action is. Both the bulk and defect interactions are classically marginal in $d=4$ dimensions, hence we look for the fixed point in an ε-expansion close to four dimensions. First, one needs to derive the β-function for the quartic coupling. Since this is a classic derivation, we do not derive it here (a good reference is [47]), but merely cite the results. Note that throughout this section, we use the following propagator:

$$G(x) = \frac{2^{d-2}\Gamma\left(\frac{d-2}{2}\right)}{(4\pi)^{\frac{d}{2}}}\frac{1}{|x|^{d-2}} = \frac{\mathcal{N}_\phi^2}{|x|^{d-2}}\,, \tag{4.3}$$

which is easily derived in momentum space before applying an inverse Fourier transform and using Schwinger parameters to compute the associated integral. At two-loop order, we also need the wavefunction renormalization of ϕ, such that $\phi = Z_\phi[\phi]$ with ϕ the bare field and $[\phi]$ the renormalized one. The relevant pole comes from the so-called 'sunset' diagram which only appears at two-loop order, and leads to

$$Z_\phi = 1 - \frac{(N+2)\,\lambda^2}{72\,(4\pi)\,\varepsilon} + O\left(\frac{\lambda^3}{(4\pi)^6}\right)\,, \tag{4.4}$$

$$\beta_\lambda = -\varepsilon\lambda + \frac{N+8}{3}\frac{\lambda^2}{(4\pi)^2} - \frac{14+3N}{3}\frac{\lambda^3}{(4\pi)^4} + O\left(\frac{\lambda^4}{(4\pi)^6}\right)\,. \tag{4.5}$$

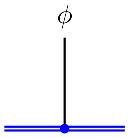
The above β-function has a non-trivial (Wilson-Fisher) fixed point at

$$\frac{\lambda_*}{(4\pi)^2} = \frac{3}{N+8}\varepsilon + \frac{6(3N+14)}{(N+8)^3}\varepsilon^2 + O\left(\varepsilon^2\right)\,. \tag{4.6}$$

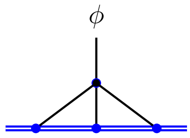
Next, the renormalization of the magnetic coupling h is due to the following diagrams, which can be systematically evaluated using master integrals in appendix A. Using the shorthand

$$w_A^{(d)} = (4\pi)^{d/2}2^{-A}\frac{\Gamma\left(\frac{d-A}{2}\right)}{\Gamma\left(\frac{A}{2}\right)}\,, \tag{4.7}$$

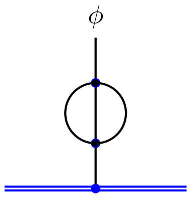
we arrive at the following expressions (note that the blue line is the defect in the Feynman diagrams below):



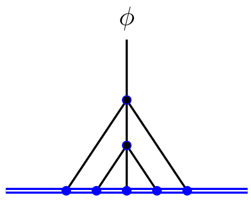
$$= -\mathcal{N}_\phi^2 h_0 \frac{\sqrt{\pi}\,\Gamma\!\left(\Delta_\phi-\frac{1}{2}\right)}{\Gamma(\Delta_\phi)} \frac{1}{|x_\perp|^{2\Delta_\phi-1}} \,, \tag{4.8}$$



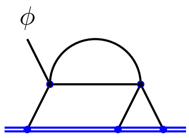
$$= \frac{\left(\mathcal{N}_\phi^2\right)^4 \lambda_0 h_0^3}{6} \left[\frac{\sqrt{\pi}\,\Gamma\!\left(\Delta_\phi-\frac{1}{2}\right)}{\Gamma(\Delta_\phi)}\right]^4 \frac{w^{(d-1)}_{2\Delta_\phi-1} w^{(d-1)}_{6\Delta_\phi-3}}{w^{(d-1)}_{8\Delta_\phi-3-d}} \frac{1}{|x_\perp|^{8\Delta_\phi-d-3}} \,, \tag{4.9}$$



$$= -\frac{(N+2)\left(\mathcal{N}_\phi^2\right)^5 \lambda_0^2 h_0}{18} \frac{\pi^{\frac{3}{2}}\Gamma\!\left(\Delta_\phi-\frac{1}{2}\right)^2\Gamma\!\left(3\Delta_\phi-\frac{1}{2}\right)}{\Gamma(\Delta_\phi)^2\Gamma(3\Delta_\phi)} \frac{w^{(d-1)}_{6\Delta_\phi-1}\left[w^{(d-1)}_{2\Delta_\phi-1}\right]^2}{w^{(d-1)}_{10\Delta_\phi-2d-1}} \frac{1}{|x_\perp|^{10\Delta_\phi-2d-1}} \,, \tag{4.10}$$



$$= -\frac{\left(\mathcal{N}_\phi^2\right)^7 \lambda_0^2 h_0^5}{12} \left[\frac{\sqrt{\pi}\,\Gamma\!\left(\Delta_\phi-\frac{1}{2}\right)}{\Gamma(\Delta_\phi)}\right]^7 \frac{\left[w^{(d-1)}_{2\Delta_\phi-1}\right]^2 w^{(d-1)}_{6\Delta_\phi-3} w^{(d-1)}_{12\Delta_\phi-d-5}}{w^{(d-1)}_{8\Delta_\phi-d-3} w^{(d-1)}_{14\Delta_\phi-2d-5}} \frac{1}{|x_\perp|^{14\Delta_\phi-2d-5}} \,, \tag{4.11}$$



$$\begin{aligned} &= -\frac{(N+8)\left(\mathcal{N}_\phi^2\right)^6 \lambda_0^2 h_0^3}{36} \frac{\Gamma\!\left(\Delta_\phi-\frac{1}{2}\right)^4 \pi^{\frac{5}{2}}\Gamma\!\left(2\Delta_\phi-\frac{1}{2}\right)}{\Gamma(\Delta_\phi)^4\Gamma(2\Delta_\phi)} \\ &\times \frac{w^{(d-1)}_{4\Delta_\phi-1} w^{(d-1)}_{4\Delta_\phi-2} w^{(d-1)}_{2\Delta_\phi-1} w^{(d-1)}_{10\Delta_\phi-d-3}}{w^{(d-1)}_{8\Delta_\phi-d-2} w^{(d-1)}_{12\Delta_\phi-2d-3}} \frac{1}{|x_\perp|^{12\Delta_\phi-2d-3}} \,. \end{aligned} \tag{4.12}$$

Renormalizing these diagrams leads to the following renormalization factor Z_h for the h coupling:

$$Z_h = 1 + \frac{h^2}{12\varepsilon}\frac{\lambda}{(4\pi)^2} + \left[\frac{(N+2)h}{72\varepsilon} + \left(\frac{1}{\varepsilon^2} - \frac{1}{\varepsilon}\right)\frac{(N+8)h^2}{108} + \left(\frac{1}{2\varepsilon^2} - \frac{1}{\varepsilon}\right)\frac{h^4}{48}\right]\frac{\lambda^2}{(4\pi)^4} + O\left(\frac{\lambda^3}{(4\pi)^6}\right) \,, \tag{4.13}$$

which subsequently yields the following β-function

$$\beta_h = -\frac{\varepsilon}{2}h + \frac{\lambda}{(4\pi)^2}\frac{h^3}{6} + \frac{\lambda^2}{(4\pi)^4}\left(\frac{N+2}{36}h - \frac{N+8}{36}h^3 - \frac{h^5}{12}\right) + O\left(\frac{\lambda^3}{(4\pi)^6}\right) \,. \tag{4.14}$$

A non trivial defect fixed point can then be derived perturbatively in ε:

$$h_*^2 = (N+8) + \frac{4N^2+45N+170}{2N+16}\varepsilon + O(\varepsilon^2) \,. \tag{4.15}$$

Interestingly enough, this has a finite non-zero value. This is essentially because at a given order in λ, there is only a finite number of allowed diagrams.

4.1.3 Some observables

Scaling dimension. Now that we know the fixed point of the full theory, we can derive some CFT data. We won't go too far in this direction however, and the reader can get a more in depth analysis in [4]. Some fairly low-hanging fruit in light of what we've done above is the defect scaling dimension of ϕ. It is given by

$$\hat{\Delta}_{\hat{\phi}} = d + \left.\frac{\partial \beta_h}{\partial h}\right|_{h=h_*,\lambda=\lambda_*} , \tag{4.16}$$

which stems from a classic result in CFT recalled in appendix B. In this particular case:

$$\hat{\Delta}_{\hat{\phi}} = 1 + \varepsilon - \varepsilon^2 \frac{3N^2 + 49N + 194}{2(N+8)^2} + O(\varepsilon^3) . \tag{4.17}$$

Interestingly enough, the defect scaling dimension of ϕ is distinct from its bulk dimension. Moreover, while the former is renormalized at one-loop, the latter only receives quantum corrections starting at two-loop order:

$$\Delta_\phi = \frac{d-2}{2} + \gamma_\phi = 1 - \frac{\varepsilon}{2} + \frac{N+2}{4(N+8)^2}\varepsilon^2 + O(\varepsilon^3) , \tag{4.18}$$

where $\gamma_\phi = d \log Z_\phi / d \log \mu$ is the (bulk) anomalous dimension of ϕ derived using its wavefunction renormalization factor and μ is an energy scale.

One-point function. The next CFT datum we consider is the one-point function of the bulk field ϕ, which was the observable we used to renormalize the defect coupling. Defining

$$\langle \phi_a(x) \rangle = \delta_{a,1} \frac{\mathcal{N}_\phi a_\phi}{|x_\perp|^{\Delta_\phi}} , \tag{4.19}$$

and inserting the renormalized coupling in the diagrams above, we find

$$a_\phi^2 = \frac{N+8}{4} + \frac{2(N+8)^2 \log 2 + N^2 - 3N - 22}{8(N+8)} \varepsilon + O\left(\varepsilon^2\right) . \tag{4.20}$$

Note that ϕ is endowed with a non-zero one-point function only along the direction of the magnetic field. This is a symptom of the initial $O(N)$ symmetry being broken to $O(N-1)$.

4.2 Surface defect in the $O(N)$ model

4.2.1 Presentation of the model

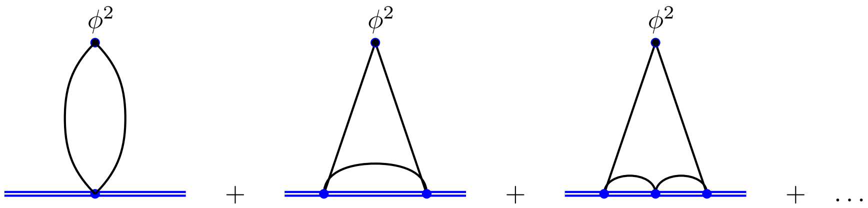
Let's study the next simplest case of a defect in the $O(N)$ model. We previously studied a line defect, and we now turn our attention to a surface defect which was extensively studied in [5, 6]:

$$S[\phi] = \frac{1}{2} \int d^d x \partial_\mu \phi_a \partial^\mu \phi_a + \frac{\lambda_0}{4!} \int d^d x \left(\phi_a(x)^2\right)^2 + h_0 \int d\tau \phi_a(\tau)^2 . \tag{4.21}$$

It's worth noting that this defect deformation resembles a mass term localized on the surface defect. This model has been used, for example, to study Ising spins in three dimensions in the presence of a boundary [48, 49]. One interesting difference with the localized magnetic field described in section 4.1 is that the defect does not break $O(N)$ symmetry.

4.2.2 ε-expansion

Again, the defect interaction is marginally relevant in $d = 4 - \varepsilon$ dimensions. Consider the one-point function of ϕ^2. In momentum space, the renormalization of the one-point function of ϕ^2 is equivalent to that of $\langle \hat{\phi}(k,m)\hat{\phi}(-k,n) \rangle$ where k is the momentum along the defect and m, n are orthogonal to the defect. The associated diagrams form a geometric sequence (this argument is taken from [6]):



$$= \sum_{n=0}^{+\infty} A_n \,, \tag{4.22}$$

where

$$\begin{cases} A_0 = -2h_0 \frac{N}{(k^2+m^2)(k^2+n^2)} \\ A_{n+1} = t A_n \end{cases} \tag{4.23}$$

with

$$t\left(k\right) = -2h_0 \int \frac{d^{d-2}p}{\left(2\pi\right)^{d-2}} \frac{1}{k^2+p^2} = -\frac{2h_0 k^{d-4}\Gamma\left(2-\frac{d}{2}\right)}{\left(4\pi\right)^{\frac{d-2}{2}}} = -\frac{h}{\pi\varepsilon} + O(1) \,. \tag{4.24}$$

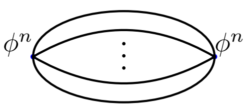
Hence we can absorb divergences by introducing a running coupling h:

$$h_0 = \mu^\varepsilon h \left(1 + \frac{h}{\pi\varepsilon} + \left(\frac{h}{\pi\varepsilon}\right)^2 + \dots\right) = \frac{\mu^\varepsilon h}{1-\frac{h}{\pi\varepsilon}} \,. \tag{4.25}$$

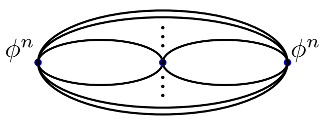
This yields the following β-function, which is exact (i.e. non-perturbative in the coupling):

$$\beta_h = -\varepsilon h + \frac{h^2}{\pi} \,, \tag{4.26}$$

and one can then easily check that the non-trivial fixed point is $h_* = \frac{\pi(1+\kappa)}{2}\varepsilon \ll 1$. Next, let's consider the interacting theory. Before renormalizing the coupling, we need to figure out the wavefunction renormalization of ϕ^2. Contrary to ϕ, ϕ^n for $n > 1$ renormalizes at one-loop via the following diagrams:



$$= \frac{Nn!\left(\mathcal{N}_\phi^2\right)^n}{|x|^{2n\Delta_\phi}} \,, \tag{4.27}$$

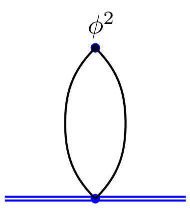


$$= -\frac{N(N+2)n!n(n-1)\left(\mathcal{N}_\phi^2\right)^{n+2}\lambda_0}{12} \frac{\left(w^{(d)}_{4\Delta_\phi}\right)^2}{w^{(d)}_{8\Delta_\phi-d}} \frac{1}{|x|^{(8+2n-4)\Delta_\phi-d}} \,. \tag{4.28}$$

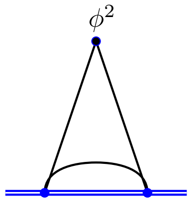
The second diagram diverges, and introducing Z_n such that $\phi^n = Z_n[\phi^n]$ leads to:

$$Z_n = 1 - \frac{(N+2)n(n-1)}{6\varepsilon} \frac{\lambda}{(4\pi)^2} + O\left(\frac{\lambda^2}{(4\pi)^4}\right) \tag{4.29}$$

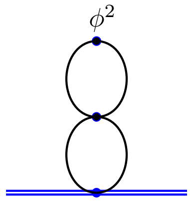
Next, the coupling h is renormalized via the following diagrams to one-loop order:



$$= -2h_0\left(\mathcal{N}_\phi^2\right)^2 N \frac{\pi}{2\Delta_\phi - 1} \frac{1}{|x_\perp|^{4\Delta_\phi-2}} \,, \tag{4.30}$$



$$= 4\left(\mathcal{N}_\phi^2\right)^3 N h_0^2 \frac{\pi^3\Gamma(2\Delta_\phi-1)^2\Gamma(3\Delta_\phi-2)}{\Gamma(\Delta_\phi)^3\Gamma(4\Delta_\phi-2)\sin(\pi\Delta_\phi)}\frac{1}{|x_\perp|^{6\Delta_\phi-4}}\,, \tag{4.31}$$



$$= \frac{h_0\lambda_0\left(\mathcal{N}_\phi^2\right)^4 N(N+2)}{3}\frac{\left[w^{(d)}_{4\Delta_\phi}\right]^2}{w^{(d)}_{8\Delta_\phi-d}}\frac{\pi}{4\Delta_\phi-1-\frac{d}{2}}\frac{1}{|x_\perp|^{8\Delta_\phi-d-2}}\,. \tag{4.32}$$

This yields the following renormalization for h at two-loops (h is perturbative):

$$h_0 = \mu^\varepsilon h\left(\frac{1}{1-\frac{h}{\pi\varepsilon}} + \frac{(N+2)\lambda}{48\pi^2\kappa\varepsilon}\right) + \mathcal{O}\left(h^3, h^2\lambda, h\lambda^2, \lambda^3\right)\,, \tag{4.33}$$

$$\beta_h = -\varepsilon h + \frac{h^2}{\pi} + \frac{N+2}{48\pi^2}h\lambda\,, \tag{4.34}$$

$$h_* = \pi\varepsilon - \frac{N+2}{48\pi}\lambda\,. \tag{4.35}$$

At the bulk fixed-point:

$$h_* = \frac{6\pi}{N+8}\varepsilon + O\left(\varepsilon^2\right)\,. \tag{4.36}$$

Contrary to the previous fixed point for the localized magnetic field, this one *is* perturbative.

4.2.3 Some observables

Scaling dimension. As before, a derivative of the beta function for h gives us access to the following defect scaling dimension at the fixed point:

$$\hat{\Delta}_{\hat{\phi}^2} = 2 + \frac{6}{N+8}\varepsilon + O(\varepsilon^2)\,. \tag{4.37}$$

One point data. The Feynman diagrams we used to renormalize the coupling also give us access to the one-point function $\langle\phi^2\rangle$:

$$a_{\phi^2} = -\frac{3}{N+8}\sqrt{\frac{N}{2}}\varepsilon + O(\varepsilon^2)\,. \tag{4.38}$$

No Feynman diagram accounts for a non-zero one-point function for ϕ. This is due to the bulk $O(N)$ symmetry being preserved by the surface defect, or more simply the lack of vertices with odd powers.

5 Conclusion

The main goal of this work was to introduce the reader to conformal defects and some of the main literature which deals with them. Prior to delving into defects exclusively, we gave a brief overview of some very useful techniques in CFT for dimensions $d > 2$, such as the embedding space formalism [1, 22], which provides a very economical way of deriving constraints on correlation functions, both with and without defects. Another vital aspect we introduced was the OPE, and more particularly the bulk-defect OPE, which paves the way towards bootstrapping defect CFTs and deriving some non-perturbative results.

A classic result in defect CFT regards the existence of non-trivial unitary conformal defects in free field theory, or rather the lack thereof. We've attempted to outline the main ideas of the proof presented in [2], and refer the reader to the original paper for a complete argument. In short, it boils down to constraining bulk-defect OPEs in order to fully characterize the operators which live on the defect, and then subsequently constraining the OPEs of the defect operators in order to demonstrate that they obey Wick's theorem. However, as mentioned previously, this proof does not apply to interacting theories – of which we've given two examples in section 4 – nor does it apply to non-local ones [45].

In the last section, we studied the localized magnetic field and the surface defect, which provide some famous examples of defects in conformal field theory. We derived their respective fixed points via diagrammatic methods, and subsequently computed some observables pertaining to scaling on the defect and one-point data. One subject we swept under the rug was boundary CFT, which can be viewed as a volume defect of sorts if working in four dimensions. Some techniques apply specifically to boundaries and simplify computations considerably, such as in the case of deriving the general form of conformal blocks in the two-point function (see [36] for a pedagogical explanation). Incidentally, our classification of allowed defect interactions in section 2.3 also showed that a cubic interaction on a boundary in scalar field theory close to four dimensions might be worth studying. This has indeed been done close to four dimensions [40], and is a natural extension to the models we studied in section 4.

A Useful integrals

The diagrams in section 4 make use of the following integrals.

Integral over a bulk vertex

$$I(x) := \int \frac{d^d y}{|x-y|^A |y|^B} = \frac{w_A^{(d)} w_B^{(d)}}{w_{A+B-d}^{(d)}} \frac{1}{|x|^{A+B-d}} \tag{A.1}$$

Integral over a defect vertex

$$\int \frac{d^p \tau}{|x - x(\tau)|^\alpha} = \frac{\pi^{\frac{p}{2}} \Gamma\left(\frac{\alpha-p}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)} \frac{1}{|x_\perp|^{\alpha-p}} . \tag{A.2}$$

Integral over a bulk vertex with defect propagator

$$\int \frac{d^d y}{|x-y|^\alpha \left|y_\parallel\right|^\beta} = \frac{\pi^{\frac{d-p}{2}} \Gamma\left(\frac{\alpha-d+p}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)} \frac{w_{\alpha-d+p}^{(p)} w_\beta^{(p)}}{w_{\alpha+\beta-d}^{(p)}} \frac{1}{\left|x_\parallel\right|^{\alpha+\beta-d}} . \tag{A.3}$$

Three propagators

$$\begin{aligned} &\int \frac{d^d k d^d l}{(k^2+m^2)^{\lambda_1} \left[(k+l)^2\right]^{\lambda_2} (l^2+m^2)^{\lambda_3}} \\ &= \frac{\pi^d \Gamma\left(\lambda_1+\lambda_2-\frac{d}{2}\right) \Gamma\left(\lambda_2+\lambda_3-\frac{d}{2}\right) \Gamma\left(\frac{d}{2}-\lambda_2\right) \Gamma\left(\lambda_1+\lambda_2+\lambda_3-d\right)}{\Gamma(\lambda_1) \Gamma(\lambda_3) \Gamma(\lambda_1+2\lambda_2+\lambda_3-d) \Gamma\left(\frac{d}{2}\right)} \frac{1}{(m^2)^{\lambda_1+\lambda_2+\lambda_3-d}} . \end{aligned} \tag{A.4}$$

B Anomalous dimension from a β-function

Let's prove that for an interaction going like $g\mathcal{O}$, we have:

$$\Delta_{\mathcal{O}} = d + \frac{\partial \beta}{\partial g}\,. \tag{B.1}$$

To prove this, we adapt arguments from [50]. We first show that $T^{\mu}{}_{\mu} = \beta_g \mathcal{O}$. Consider an infinitesimal scale transformation $x \mapsto (1+\varepsilon)x$. In QFT, this also affects the coupling g. Since energy goes like inverse length, we have:

$$g(\mu) \mapsto g\left(\frac{\mu}{1+\varepsilon}\right) = g(\mu) - \varepsilon \beta_g\,. \tag{B.2}$$

We then require that the action remain unchanged during such a transformation:

$$\delta S = \varepsilon \int d^d x \left[\partial_\mu j^\mu - \frac{\partial \mathcal{L}}{\partial g}\beta_g\right]\,, \tag{B.3}$$

where the first term is the classical variation of the action, expressed using the dilatation current $j^\mu = T^{\mu}{}_{\nu} x^\nu$ where $T^{\mu\nu}$ is the stress tensor, and the second is the quantum variation. We've supposed that $\partial\mathcal{L}/\partial g = \mathcal{O}$, hence

$$\partial_\mu j^\mu - \frac{\partial \mathcal{L}}{\partial g}\beta_g = \left(\partial_\mu T^{\mu}{}_{\nu}\right) x^\nu + T^{\mu}{}_{\mu} - \beta_g \mathcal{O} = 0\,. \tag{B.4}$$

Next, since one can write $dO/d\log\mu = -\Delta_O O$ for any operator O and using that $\Delta_{T_{\mu\nu}} = d$ is protected, we get, on the one hand,

$$\frac{d}{d\log\mu} T^{\mu}{}_{\mu} = -d T^{\mu}{}_{\mu} = -d\beta_g \mathcal{O}\,, \tag{B.5}$$

and on the other:

$$\frac{d}{d\log\mu} T^{\mu}{}_{\mu} = \frac{\partial \beta_g}{\partial g}\beta_g \mathcal{O} - \beta_g \Delta_{\mathcal{O}} \mathcal{O}\,. \tag{B.6}$$

The RHS of the last two equalities being equal to one another leads to the desired result.

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